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NUMERICAL SOLUTION OF FREE SURFACE, POROUS FLOW PROBLEMS

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SUMMARY

An inexpensive, finite difference numerical method is developed for the approximate solution of general, free surface, porous flow problems. The method is so designed that the required numerical boundary conditions coincide exactly with the required physical boundary conditions. In the present paper, application is made to prototype, steady state, dam flow problems.

KEY WORDS Navier–Stokes Free Surface Porous Flow

1. INTRODUCTION

Porous flow is of interest, for example, to mathematicians, geologists, ecologists, civil engineers and fluid dynamicists (see, e.g., References 1, 9, 11 and the numerous additional reverences contained therein). Models are usually,[11] but not always,[8] formulated in terms of classical continuum mechanics, with equations of motion often being highly non-linear. Related problems are further complicated by the presence of a free boundary, and most modern computer methodology has been applicable only under highly restrictive assumptions.[5,7,10,11,13]

In this paper we will develop a method for general, continuum formulated, free surface, porous flow problems. The method is analogous to that developed recently by Casulli[3], who showed that combining the Los Alamos MAC method for Navier–Stokes problems[10,13] with the Courant–Isaacson–Rees method for hyperbolic systems[6] leads to a new method which is more powerful than either of its constituents. This new method allows for thermal, non-homogeneous and three-dimensional considerations.

2. GOVERNING EQUATIONS

For *simplicity* only, let us consider the flow of a *homogeneous* fluid in a porous medium. The governing three-dimensional equations are derived from the principles of conservation of momentum and mass, and are given as follows.[14] The generalized Darcy's equation of motion is

$$\frac{1}{\eta}\frac{\mathrm{D}u_i}{\mathrm{D}t}=-\frac{\partial\phi}{\partial x_i}+X_i-\frac{\nu}{k}u_i, \tag{1}$$

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where η is the porosity of the medium, $\mathbf{u}$ is the seepage fluid velocity, ϕ is the ratio of pressure to the constant density of the fluid, $\mathbf{X}$ is a body force, ν is the kinematic viscosity coefficient, and k is the permeability of the medium. The equation of continuity for an incompressible fluid is

$$\frac{\partial u_\alpha}{\partial x_\alpha}=0, \tag{2}$$

where, as usual, repetition of indices denotes summation. For any free surface which can be represented as a single valued function $x_3=h(x_1, x_2, t)$ the change in surface elevation is determined[11] by the kinematic condition

$$\eta_s\frac{\partial h}{\partial t}+u_{1,s}\frac{\partial h}{\partial x_1}+u_{2,s}\frac{\partial h}{\partial x_2}=u_{3,s}, \tag{3}$$

where η_s is the porosity of the medium at the free surface, and where $\mathbf{u}_s$ is the seepage velocity at the fluid surface.

The governing two-dimensional equations for the models most often considered can now be derived from system (1)–(3) as follows. The temporal acceleration term in Darcy's law, as stressed by Yih (Reference 14, p. 277) must be considered in the formulation of transient problems, so that it has not been retained to allow a false transient numerical procedure to be used. However, convective acceleration in (1) is relatively insignificant and can be neglected. Thus, in the usual notation, we arrive at

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}=-\eta\frac{\partial \phi}{\partial x}-\frac{\eta\nu}{k}u\\&\frac{\partial v}{\partial t}=-\eta\frac{\partial \phi}{\partial y}-\frac{\eta\nu}{k}v-g\eta\\&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\end{aligned}\right\} \tag{4}$$

with the free surface equation

$$\eta_s\frac{\partial h}{\partial t}+u_s\frac{\partial h}{\partial x}=v_s. \tag{5}$$

In applying (4) and (5), the following initial conditions are employed:

$$\left.\begin{aligned}u(x, y, t_0)&=u_0(x, y)\\v(x, y, t_0)&=v_0(x, y)\\h(x, t_0)&=h_0(x)\end{aligned}\right\}. \tag{6}$$

The boundary conditions for (4) are given by assigning either the pressure ϕ_s or the normal velocity $u_\perp$ as a function of time on the boundary. In particular, at the free surface the pressure is zero at all times, while at the impenetrable boundaries the normal velocity is zero.

The free surface equation (5) needs one boundary condition at the point of intersection with the fixed boundary only if the fixed boundary is an inflow boundary at such a point.[2] In this case the surface height $h(x, t)$ must be specified at this point as a function of time.

Further consideration about boundary conditions will be given in Section 5.

3. DIFFERENCE APPROXIMATIONS FOR THE EQUATIONS OF MOTION

Our finite approximations for system (4) are based on the MAC (Marker-and-Cell) method[10,13] and can be extended, if desired, in a direct way to system (1)–(2). Specifically, the finite difference mesh consists of rectangular cells of width Δx and height Δy. The field variables u, v and ϕ are defined at the locations shown in Figure 1: the u component of velocity is defined at the centre of each vertical side of a cell, the v component of velocity at the centre of each horizontal side, and the pressure ϕ at each cell centre.

The finite difference equations corresponding to the first two equations of system (4), that is, to Darcy's equations, are

$$\begin{aligned} u^{n+1}_{i+\frac{1}{2},j} &= u^{n}_{i+\frac{1}{2},j} - \Delta t\left[\eta_{i+\frac{1}{2},j}\frac{\phi^{n+\frac{1}{2}}_{i+1,j}-\phi^{n+\frac{1}{2}}_{i,j}}{\Delta x} + \left(\frac{\eta\nu}{k}\right)_{i+\frac{1}{2},j} u^{n}_{i+\frac{1}{2},j}\right] \\ v^{n+1}_{i,j+\frac{1}{2}} &= v^{n}_{i,j+\frac{1}{2}} - \Delta t\left[\eta_{i,j+\frac{1}{2}}\frac{\phi^{n+\frac{1}{2}}_{i,j+1}-\phi^{n+\frac{1}{2}}_{i,j}}{\Delta y} + \left(\frac{\eta\nu}{k}\right)_{i,j+\frac{1}{2}} v^{n}_{i,j+\frac{1}{2}} + g\eta_{i,j+\frac{1}{2}}\right]. \end{aligned} \tag{7}$$

Of course, implementation of (7) requires first the determination of the pressure field $\phi^{n+\frac{1}{2}}_{i,j}$, which is accomplished as follows.

The finite difference equation corresponding to the last equation of system (4), that is, the incompressibility condition, in each cell is given by

$$\frac{u^{n+1}_{i+\frac{1}{2},j}-u^{n+1}_{i-\frac{1}{2},j}}{\Delta x} + \frac{v^{n+1}_{i,j+\frac{1}{2}}-v^{n+1}_{i,j-\frac{1}{2}}}{\Delta y} = 0. \tag{8}$$

As in Reference 3, the pressure field $\phi^{n+\frac{1}{2}}_{i,j}$ at the centre of each cell must be computed in such a way that the discrete incompressibility condition (8) is valid throughout. To find a difference equation for the pressure, we require that the seepage velocity components computed with (7) satisfy (8). We first define

$$\begin{aligned} F^{n}_{i+\frac{1}{2},j} &= \left[1-\Delta t\left(\frac{\eta\nu}{k}\right)_{i+\frac{1}{2},j}\right]u^{n}_{i+\frac{1}{2},j} \\ G^{n}_{i,j+\frac{1}{2}} &= \left[1-\Delta t\left(\frac{\eta\nu}{k}\right)_{i,j+\frac{1}{2}}\right]v^{n}_{i,j+\frac{1}{2}} - (\Delta t)g\eta_{i,j+\frac{1}{2}}, \end{aligned}$$

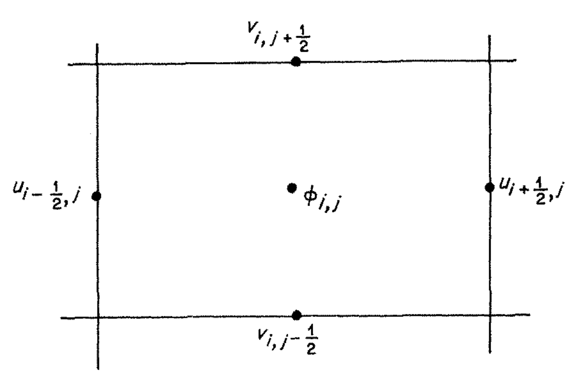

Figure 1. Positions of field variables

so that (7) become

$$u^{n+1}_{i+\frac{1}{2},j} = F^n_{i+\frac{1}{2},j} - (\Delta t)\eta_{i+\frac{1}{2},j}\frac{\phi^{n+\frac{1}{2}}_{i+1,j} - \phi^{n+\frac{1}{2}}_{i,j}}{\Delta x}$$
$$v^{n+1}_{i,j+\frac{1}{2}} = G^n_{i,j+\frac{1}{2}} - (\Delta t)\eta_{i,j+\frac{1}{2}}\frac{\phi^{n+\frac{1}{2}}_{i,j+1} - \phi^{n+\frac{1}{2}}_{i,j}}{\Delta y}. \tag{9}$$

Substitution of (9) into (8) yields the following finite difference equation for pressure:

$$\eta_{i+\frac{1}{2},j}\frac{\phi^{n+\frac{1}{2}}_{i+1,j} - \phi^{n+\frac{1}{2}}_{i,j}}{(\Delta x)^2} - \eta_{i-\frac{1}{2},j}\frac{\phi^{n+\frac{1}{2}}_{i,j} - \phi^{n+\frac{1}{2}}_{i-1,j}}{(\Delta x)^2} + \eta_{i,j+\frac{1}{2}}\frac{\phi^{n+\frac{1}{2}}_{i,j+1} - \phi^{n+\frac{1}{2}}_{i,j}}{(\Delta y)^2} - \eta_{i,j-\frac{1}{2}}\frac{\phi^{n+\frac{1}{2}}_{i,j} - \phi^{n+\frac{1}{2}}_{i,j-1}}{(\Delta y)^2}$$
$$= \frac{F^n_{i+\frac{1}{2},j} - F^n_{i-\frac{1}{2},j}}{(\Delta t)(\Delta x)} + \frac{G^n_{i,j+\frac{1}{2}} - G^n_{i,j-\frac{1}{2}}}{(\Delta t)(\Delta y)}. \tag{10}$$

Equation (10) is then solved at each time step by iteration.

Note that for constant porosity η, equation (10) reduces to a finite difference Poisson equation.

4. APPROXIMATION OF FREE SURFACE MOTION

A finite difference scheme which discretizes the free surface equation (5) and simultaneously allows for the correct physical boundary conditions[2] is the Courant, Isaacson and Rees method, which is implemented as follows. The surface height $h^n_{i+\frac{1}{2}}$ is defined on each vertical grid line and the equation (5) is approximated by

$$h^{n+1}_{i+\frac{1}{2}} = h^n_{i+\frac{1}{2}} + \frac{\Delta t}{\eta_{s,i+\frac{1}{2}}}\left[v^n_s - u^n_s\frac{\Delta h^n_{i+\frac{1}{2}}}{\Delta x}\right], \tag{11}$$

where

$$\Delta h^n_{i+\frac{1}{2}} = h^n_{i+\frac{1}{2}} - h^n_{i-\frac{1}{2}}, \quad \text{if} \quad u^n_s \geq 0$$
$$\Delta h^n_{i+\frac{1}{2}} = h^n_{i+\frac{3}{2}} - h^n_{i+\frac{1}{2}}, \quad \text{if} \quad u^n_s < 0.$$

The coefficients u^n_s and v^n_s in (11) are obtained as weighted averages of the nearest cell velocities.

5. BOUNDARY CONDITIONS

To find seepage velocities with (7), a knowledge of the pressure field $\phi^{n+\frac{1}{2}}_{i,j}$ is required at each cell centre, *including* the boundary cells, which are crossed by boundary lines. For very general boundary configurations and conditions, then, the following special considerations will be applied to boundary cells.

When the pressure ϕ_s is specified as a boundary condition, the pressure $\phi^{n+\frac{1}{2}}_{i,j}$ at the boundary cell centre is chosen such that a linear interpolation between it and the pressure in the nearest cell yields the boundary pressure ϕ_s. As an example, the pressure $\phi^{n+\frac{1}{2}}_{i,j}$ for the situation illustrated in Figure 2 is given by

$$\phi^{n+\frac{1}{2}}_{i,j} = (1-q)\phi^{n+\frac{1}{2}}_{i,j-1} + q\phi_s, \tag{12}$$

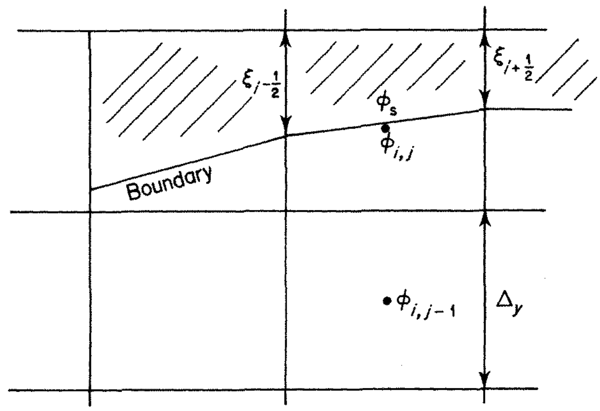

Figure 2. Boundary configuration

where

$$q = \frac{2\Delta y}{3\Delta y - (\xi_{i-\frac{1}{2}} + \xi_{i+\frac{1}{2}})}.$$

Particular care must be taken also where the normal seepage $u_\perp$ is prescribed on a boundary. In this connection, note that (8) can be rewritten as

$$u^{n+1}_{i+\frac{1}{2},j}\Delta y - u^{n+1}_{i-\frac{1}{2},j}\Delta y + v^{n+1}_{i,j+\frac{1}{2}}\Delta x - v^{n+1}_{i,j-\frac{1}{2}}\Delta x = 0, \tag{13}$$

which has the following physical meaning: the volume of fluid entering cell (i, j) must balance the volume of fluid leaving the cell. It is this principle which is applied to a cell crossed by the boundary. With reference to Figure 3, where $u_\perp$ denotes the normal velocity at which the fluid enters cell (i, j) through the boundary, the analogue of (13) in such a cell is given by

$$u^{n+1}_{i+\frac{1}{2},j}\xi_{i+\frac{1}{2}} - u^{n+1}_{i-\frac{1}{2},j}\xi_{i-\frac{1}{2}} + v^{n+1}_{i,j+\frac{1}{2}}\Delta x - u^{n+1}_\perp((\Delta x)^2 + (\xi_{i-\frac{1}{2}} - \xi_{i+\frac{1}{2}})^2)^{\frac{1}{2}} = 0. \tag{14}$$

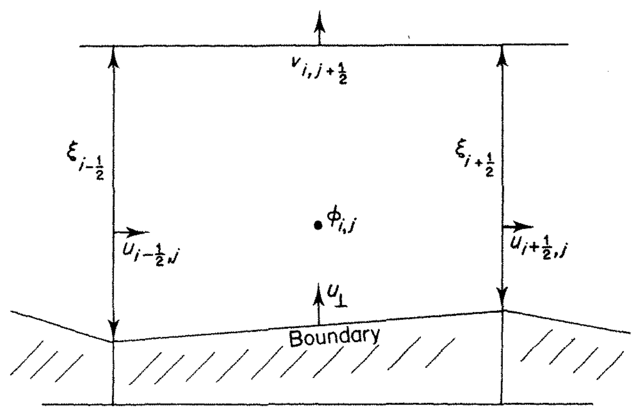

Figure 3. Boundary configuration

The finite difference equation for pressure in a boundary cell is obtained by substituting (9) into (14):

$$\eta_{i+\frac{1}{2},j}\frac{\phi_{i+1,j}^{n+\frac{1}{2}}-\phi_{i,j}^{n+\frac{1}{2}}}{\Delta x}(\xi_{i+\frac{1}{2}})-\eta_{i-\frac{1}{2},j}\frac{\phi_{i,j}^{n+\frac{1}{2}}-\phi_{i-1,j}^{n+\frac{1}{2}}}{\Delta x}(\xi_{i-\frac{1}{2}})+\eta_{i,j+\frac{1}{2}}\frac{\phi_{i,j+1}^{n+\frac{1}{2}}-\phi_{i,j}^{n+\frac{1}{2}}}{\Delta y}\Delta x$$
$$=\frac{F_{i+\frac{1}{2},j}^{n}\xi_{i+\frac{1}{2}}-F_{i-\frac{1}{2},j}^{n}\xi_{i-\frac{1}{2}}+G_{i,j+\frac{1}{2}}^{n}\Delta x}{\Delta t}+u_{\perp}^{n+1}\frac{((\Delta x)^2+(\xi_{i-\frac{1}{2}}-\xi_{i+\frac{1}{2}})^2)^{\frac{1}{2}}}{\Delta t}. \quad (15)$$

The set of equations (10) and (12) or (15) for the boundary cells form a linear system of equations for the unknowns $\phi_{i,j}^{n+\frac{1}{2}}$.

With regard to free surface equation (11), note that the term $\Delta h_{i+\frac{1}{2}}^{n}$ is always defined at the boundary *except* when such a boundary is an inflow boundary.[2] Hence, in this case, the free surface height has to be given as a function of time.

6. STABILITY AND ACCURACY CONSIDERATIONS.

The finite difference equations (7) and (10) are second-order accurate in space and first order in time, while (11) is first-order accurate in space and in time. Note that since the incompressibility conditions (8) and (14) are satisfied, the numerical scheme conserves regorously the fluid volume in each cell.

7. COMPUTER ALGORITHM AND EXAMPLES.

The algorithm for proceeding from time step t_n to time t_{n+1} can be described easily in three steps. *Step* 1: Determine intermediate values of seepage velocities by explicit calculations with (7), using the pressure and velocities at t_n. *Step* 2: Determine the pressure field so that incompressibility condition (8) or (14) is satisfied in each cell. This stage is done iteratively by calculating a pressure change with which (8) and (14) will be satisfied and then adjusting the velocities.[3] As a starting point of the iterative procedure, the intermediate values are taken from Step 1. *Step* 3: Determine the new free surface positions by an explicit calculation with (11).

For the interested reader, a typical FORTRAN program is provided in the Appendix of Reference 4.

Let us consider now a prototype, non-trivial problem of broad general interest. Consider the earth dam shown in Figure 4. The two-dimensional flow domain is ABCDE. In such a domain, the fluid flow is described by system (4) and the free surface, represented by curve BC, is described by equation (5). The fluid levels d_a and d_b are fixed so that the positions of A, B, D and E are known. The problem is to determine the free surface curve $h(x, t)$ and the point C. Thus, the problem is, in fact, a steady state problem.

As a particular case, let $d_a = 1{\cdot}8$, $d_b = 0{\cdot}6$, B = (0·9, 1·8), D = (3·4, 0·6), E = (4·0, 0·0), M = (1·0, 2·0), N = (2·0, 2·0), $\eta = 0{\cdot}1$, $\frac{\nu}{k} = g = 1{\cdot}0$. At $t_0 = 0$, the initial velocity components are $u_0 = v_0 = 0$. The boundary conditions are

$$\begin{aligned}&\phi = 0, \quad \text{on} \quad BCD\\ &\phi = g(d_a - y), \quad \text{on} \quad AB\\ &\phi = g(d_b - y), \quad \text{on} \quad DE\\ &v = 0, \quad \text{on} \quad AE.\end{aligned}$$

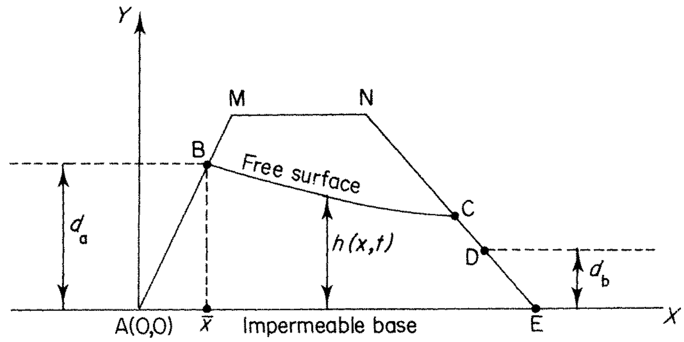

Figure 4. Schematic diagram of an earth dam configuration

Since the boundary AB will be an inflow boundary while CDE will be an outflow boundary, the boundary condition for free surface equation (5) is given simply be

$$h(\bar{x}, t) = d_a, \qquad o \leq t.$$

The numerical method was applied with $\Delta x = 0{\cdot}1$ and $\Delta y = 0{\cdot}2$, and by enclosing the configuration in a rectangle with vertices (0, 0), (4, 0), (4, 2·6), (0, 2·6), which yields a grid of 40 by 13 cells. The time step was $\Delta t = 0{\cdot}05$. In our first example the initial guess of the free surface is a straight line from B to D. Whenever the method yielded an h which was above boundary segment NE, the value of h was reset to the height of NE at the same x value, which is physically correct. Figure 5 shows the free surface positions at the times $t = 0, 1, 2, 3, 10$. At $t = 10$, the free surface was no longer changing, and is, then, the desired steady state solution. In our second example, the initial free surface configuration is the horizontal straight line through B. Figure 6 shows the free surface positions, again, at times $t = 0, 1, 2, 3, 10$. As expected, the steady state configuration at $t = 10$ is identical with that of the first example.

The running time on the IBM–370 at the University of Texas for each example described above was only 1 minute 25 seconds.

A variety of computer examples for simpler problems, in which the sides of the dam were vertical, yielded results which were identical to those which had been obtained previously by others.[1]

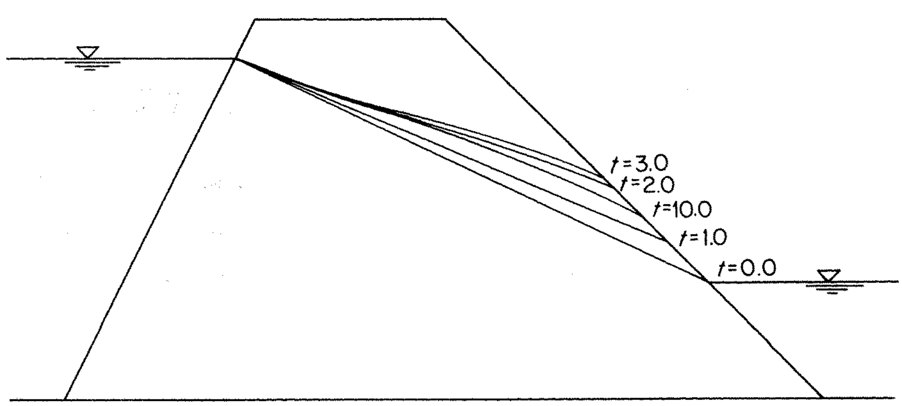

Figure 5

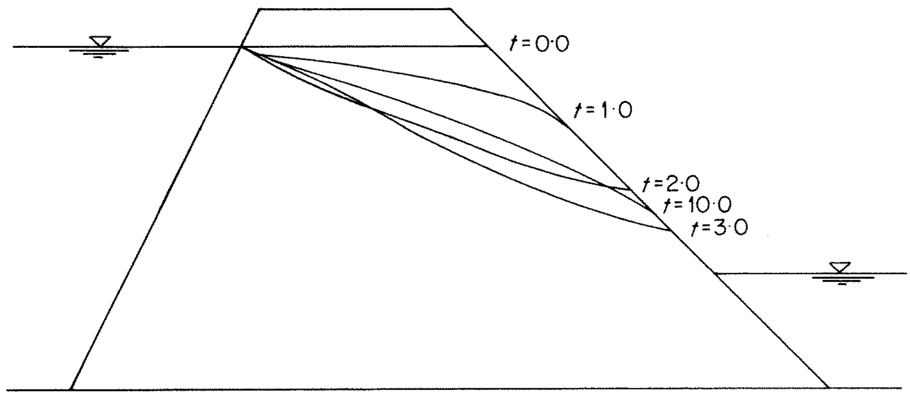

Figure 6

8. CONCLUSION.

The method presented in this paper is a new, finite difference method for free surface, porous flow problems. The method is fast, economical and accurate. It applies to general boundary configurations and conserves rigorously the fluid volume in each cell. Application in the present paper has been to a class of steady state problems. Application in a forthcoming paper will be to transient, porous, two-fluid flow for both the miscible and immiscible cases.

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